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SOME PROPERTIES OF MULTI-FLUID STOKES FLOWS

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Abstract—The solution of Stokes' equations for a rotating axisymmetric body which possesses reflection symmetry about a planar interface between two infinite immiscible quiescent viscous fluids is shown to be independent of the viscosities of the fluids and identical with the solution when the fluids have the same viscosity. The result is generalized to a rotating axisymmetric system of bodies which possesses reflection symmetry about each interface of a plane stratified system of fluids. An analogous result for two-fluid systems with a nonplanar static interface is also derived. The effect on torque reduction produced by the presence of a second fluid layer adjacent to a rotating axisymmetric body is considered and explicit calculations are given for the case of a sphere. A proof of uniqueness for unbounded multi-fluid Stokes' flow is given and the asymptotic far field structure of the velocity field is determined for axisymmetric flow caused by the rotation of axisymmetric bodies.

1. INTRODUCTION

In a recent paper, Schneider, O'Neill & Brenner (1973) have considered the slow rotation of an axisymmetrical body. The surface of the body is formed from two intersecting spheres whose circle of intersection lies in the plane of the interface between two immiscible fluids. The axis of rotation of the body is its axis of symmetry. It was established in that paper that when the body possesses reflection symmetry about the plane of the interface between the fluids, the Stokes velocity field in either of the fluids is the same as if the body were rotating in an infinite homogeneous fluid. Thus for such a body, the velocity field generated in each of the fluids is independent of the viscosities and the torque acting on the body is then proportional to the sum of the viscosities of the fluids.

In this paper, we show that such results occur for an axisymmetrical body of arbitrary shape which possesses reflection symmetry about the interface of two immiscible fluids. Furthermore, we show that the results can be generalized to include any axisymmetric system of solid bodies which has reflection symmetry about the plane of the interface between two fluids. A further extension can be made to an axisymmetric system of bodies which rotate in plane stratified layers of fluids, provided that reflection symmetry of the system of bodies exists about each of the fluid–fluid interfaces which we suppose are perpendicular to the axis of rotation. In all cases it is assumed that the bodies rotate with the same angular velocity and that the fluid motions may be regarded as Stokes flows.

An interesting feature of the class of flows described above is that the planar interfaces between the fluids are unstressed so that they are quasi-free surfaces and the flow generated within the different fluids are uncoupled in the sense that mechanical energy is not communicated from one fluid to another across the interfaces. It does not appear to be possible for multi-fluid systems to have a non-planar stress free interface although there is a class of flows in which a non-planar interface can be static. The effect on torque reduction produced by a second fluid layer adjacent to a rotating axisymmetric body is considered and explicit calculations are given when the body is a sphere.

To establish that the solution to a multi-fluid Stokes' flow can be derived from the corresponding solution for a homogeneous fluid, it is necessary to establish the uniqueness of Stokes' flows of multi-fluid systems. Such proofs do not appear to be available in the literature, although proofs exist for the uniqueness of Stokes' flows of a homogeneous incompressible fluid which is bounded or for the streaming of an unbounded fluid past a finite body. These proofs are given, for instance, by Finn & Noll (1957) and Ladyzhenskaya (1963). In this paper we give a short proof of uniqueness of the solution of the Stokes' equations appropriate to the axisymmetric flow caused by a rotating body or finite system of bodies in a quiescent system of unbounded immiscible fluids. We also show that if only one of the fluids is unbounded, the velocity decays to zero as the inverse square of the distance from the bodies.

2. UNIQUENESS THEOREM FOR A MULTI-FLUID STOKES' FLOW

We consider an unbounded axially symmetric multi-fluid system composed of immiscible incompressible viscous fluids which occupy the regions $\tau^{(k)}(k = 1, 2, \ldots)$, with $\sigma^{(k)}$ denoting the interface or interfaces with adjacent fluids. The motion of the fluids is caused by the slow rotation of a system of solid boundaries $S^{(m)}$, $(m = 1, 2, \ldots, M)$ about their common axis of symmetry with angular velocity $\Omega^{(m)}$.

If $(\mathbf{v}, p)$ denote the velocity and pressure fields at any point of the fluids, the equations governing the flow are

$$\left.\begin{aligned} \nabla p &= \mu \nabla^2 \mathbf{v} \\ \nabla \cdot \mathbf{v} &= 0 \end{aligned}\right\} \qquad [1]$$

within any region $\tau^{(k)}$, together with the boundary conditions

(a) $\mathbf{v} = \Omega^{(m)} r \sin\theta$ on $S^{(m)}$,

(b) continuity of $\mathbf{v}$ and $\mathbf{R}_n$, the stress vector associated with the direction $\mathbf{n}$, across $\sigma^{(k)}$. So far as the latter condition is concerned, we adopt the convention that the direction of $\mathbf{n}$, the normal to the interface, is common for adjacent fluids,

(c) $|\mathbf{v}| \to 0$ as $r \to \infty$, with (r, θ, ϕ) denoting spherical polar coordinates with $\theta = 0, \pi$ the axis of symmetry.

Equations [1] are satisfied by

$$\mathbf{v} = v(r, \theta)\hat{\boldsymbol{\phi}}, \qquad p = 0 \qquad [2]$$

within any region $\tau^{(k)}$ provided that in each such region,

$$\nabla^2 v = \frac{v}{r^2 \sin^2\theta} \qquad [3]$$

with ∇^2 denoting the axisymmetric Laplace operator and $\hat{\boldsymbol{\phi}}$ is the constant unit vector perpendicular to the azimuthal plane. If we assume that there exists a solution to the multi-fluid problem of the form [2], then the boundary conditions to be satisfied are

$$v = \Omega^{(m)} r \sin\theta$$

on any body $S^{(m)}$, where the values of $\Omega^{(m)}(m = 1, 2, \ldots, M)$ may be different, and that v and $\mu\partial v/\partial n$ both be continuous across any interface $\sigma^{(k)}$. The solution must also be regular at points of the fluid on the axis $\theta = 0, \pi$ (in fact v must vanish on this axis) and also vanish at $r = \infty$.

The uniqueness of the solution of this boundary value problem can be established by postulating two such solutions. If $u(r, \theta)$ is the difference of the two solutions, then $u(r, \theta)$ is a function satisfying [3] and vanishing on all $S^{(m)}$, the axis of symmetry, and at infinity. If $u(r, \theta)$ is not identically zero throughout the fluids, then it has a positive maximum and/or a negative minimum. However if $u(r, \theta)$ has a positive maximum, this occurs at some point A which cannot lie on any body $S^{(m)}$ or the axis of symmetry and by the maximum principle (Garabedian 1964) $u(r, \theta)$ cannot have a positive maximum at an interior point of any fluid, since [3] must be satisfied. Thus A must be on an interface. However u and $\mu\partial u/\partial n$ are continuous at an interface and since $\mu > 0$ for all fluids, it follows that $\partial u/\partial n$ vanishes at the maximum point A. Thus $\partial u/\partial n$ is continuous at A and hence the existence of $\partial^2 u/\partial n^2$ at A depends on the limits of this function being the same as A is approached from either side of the interface, i.e. have a removable discontinuity. Now the continuity of u at an interface implies that the same is true of tangential derivatives. Moreover [3] is satisfied by u at A. Consequently a positive maximum at A violates the maximum principle, giving a contradiction. Similarly $u(r, \theta)$ cannot attain a negative minimum, hence implying that $u(r, \theta) = 0$ everywhere, which establishes the uniqueness theorem.

3. ROTATION OF AN AXIALLY SYMMETRIC SYSTEM OF BODIES; THE FAR FIELD SOLUTION

We consider the case of an axially symmetric system of finite bodies which slowly rotates with constant angular velocity Ω about its axis of symmetry in a finite number of incompressible, quiescent, immiscible fluids, only one of which is unbounded. Choosing spherical polar coordinates (r, θ, ϕ) such that the axis of symmetry is the axis $\theta = 0, \pi$, there evidently exists an r_0 such that for $r > r_0$ the fluid is homogeneous and all the bodies lie within the sphere $r < r_0$.

Our attention will be confined to the unbounded region $r > r_0$, in which the equations governing the flow are [1] and are satisfied by $\mathbf{v} = v(r, \theta)\hat{\phi}$ and $p = 0$ provided that

$$L^2[v] \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\mathscr{L}[v] = 0 \qquad [4]$$

where

$$\mathscr{L}[v] \equiv \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) - \frac{v}{\sin\theta}. \qquad [5]$$

Assuming the existence of a solution, we require that $v \to 0$ as $r \to \infty$ and, for a regular solution on the axis of symmetry, it is necessary that $v = 0$ when $\theta = 0, \pi$. The boundary conditions on the bodies and at the fluid interfaces do not enter explicitly here, where it will be shown that the form of the far field solution is *necessarily* that of the solution in separated variables.

The unique bounded solution in $[0, \pi]$ of

$$\mathscr{L}[y] + n(n+1)y\sin\theta = 0, \qquad [6]$$

where n is a positive integer, is the associated Legendre function $P_n^1(\cos\theta)$ and furthermore, from Jeffreys & Jeffreys (1950), the set of associated Legendre functions $\{P_n^1(\cos\theta), n \geq 1\}$ form a complete set in $[0, \pi]$ and are orthogonal with weight function $\sin\theta$. Thus for any $r > r_0$, we can write

$$v(r, \theta) = \sum_{n=1}^{\infty} A_n(r)P_n^1(\cos\theta). \qquad [7]$$

Since differentiation term by term of this infinite series is not necessarily justified, the coefficients $\{A_n(r)\}$ are found by considering Fourier components (with respect to the orthogonal functions) of [4]. Now the self-adjointness of the operator $\mathscr{L}$, defined by [5], implies that

$$\int_0^\pi \mathscr{L}[v]P_n^1(\cos\theta)\,d\theta = \int_0^\pi v\mathscr{L}[P_n^1(\cos\theta)]\,d\theta$$
$$= -n(n+1)\int_0^\pi v(r, \theta)P_n^1(\cos\theta)\sin\theta\,d\theta$$

from [6]. Hence

$$\int_0^\pi L^2[v]P_n^1(\cos\theta)\sin\theta\,d\theta = \left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{n(n+1)}{r^2}\right\}\int_0^\pi v(r,\theta)P_n^1(\cos\theta)\sin\theta\,d\theta.$$

But the left hand side vanishes by virtue of [4] and hence the coefficients in [7] satisfy the differential equations:

$$r^2A_n''(r) + 2rA_n'(r) - n(n+1)A_n(r) = 0 \qquad (n \geq 1). \qquad [8]$$

Further, since $v(r, \theta) \to 0$ as $r \to \infty$ for all θ, the appropriate solution of [8], for each $n \geq 1$, is $A_n(r) = \Omega a_n r^{-(n+1)}$, where a_n is constant. Consequently the form of solution for $v(r, \theta)$ which satisfies the conditions at infinity and along the axis $\theta = 0, \pi$, is given by

$$v(r,\theta) = \Omega \sum_{n=1}^{\infty} a_n r^{-(n+1)} P_n^1(\cos\theta). \qquad [9]$$

This is an exact solution of [3], valid for $r > r_0, 0 \leq \theta \leq \pi$, the coefficients $\{a_n\}$ depending on the geometry of the bodies and interfaces and the relative viscosities of the fluids. In particular if the body is the sphere $r = a$ and there is only one fluid, then $a_1 = a^3, a_n = 0$ $(n \geq 2)$. In any case, [9] implies that

$$v = O(r^{-2}) \quad \text{as} \quad r \to \infty.$$

Finn & Noll (1957) established the uniqueness of three dimensional Stokes streaming flow of a homogeneous fluid past an arbitrarily shaped finite body by showing that the vorticity vector is of order r^{-2} as $r \to \infty$. In the axisymmetric case, this order of magnitude can be obtained simply by the above method since the velocity is then of the form

$$\mathbf{v} = \text{curl}\{f(r,\theta)\hat{\boldsymbol{\phi}}\}.$$

The Stokes equations imply that

$$L^4[f] = 0,$$

where the operator L^2 is given by [4], whence in the expansion

$$f(r,\theta) = \sum_{n=1}^{\infty} B_n(r)P_n^1(\cos\theta),$$

the coefficients must satisfy, for each $n \geq 1$:

$$\left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) - \frac{n(n+1)}{r^2}\right\}^2 B_n(r) = 0.$$

The leading term in f at large r is then $C_1 \sin\theta$, with C_1 a constant, and since

$$\text{curl}^2(\sin\theta\hat{\boldsymbol{\phi}}) = \frac{2}{r^2}\sin\theta\hat{\boldsymbol{\phi}},$$

the result follows.

4. ROTATING AXIALLY SYMMETRIC BODY STRADDLING THE INTERFACE BETWEEN IMMISCIBLE FLUIDS

We suppose that an axially symmetric body slowly rotates with constant angular velocity about its axis of symmetry which is perpendicular to the planar interface between two semi-infinite incompressible and immiscible fluids which are at rest at an infinite distance from the body. Taking the interface to be in the plane $z = 0$ and letting μ_1 and μ_2 denote the viscosities of the fluids in the half-spaces $z > 0$ and $z < 0$ respectively, the equations which govern the flows in the two fluids are that

$$\nabla p^{(i)} = \mu^{(i)}\nabla^2 \mathbf{v}^{(i)} \tag{10}$$

in the fluid with viscosity $\mu^{(i)}(i = 1, 2)$, together with the boundary conditions

$$\mathbf{v}^{(i)} = \Omega r \sin\theta \hat{\boldsymbol{\phi}}$$

on the body, $|\mathbf{v}^{(i)}| \to 0$ as $r \to \infty$ and the continuity conditions

$$\mathbf{v}^{(1)} = \mathbf{v}^{(2)}, \mathbf{R}_n^{(1)} = \mathbf{R}_n^{(2)} \tag{11}$$

on the interface $z = 0$. Equations [10] and [11] are satisfied by

$$p^{(i)} = 0, \mathbf{v}^{(i)} = v^{(i)}(r, \theta)\hat{\phi}, \tag{12}$$

where

$$L^2[v^{(i)}] = 0 \tag{13}$$

and

$$v^{(1)} = v^{(2)}, \qquad \frac{\mu^{(1)}}{r}\frac{\partial v^{(1)}}{\partial \theta} = \frac{\mu^{(2)}}{r}\frac{\partial v^{(2)}}{\partial \theta} \qquad \left(\theta = \frac{\pi}{2}\right). \tag{14}$$

The analysis of the preceding sections has established that there is a unique Stokes flow specified by [12] and [13]. However, if the body straddles the interface symmetrically, then [13] and [14] are satisfied when $v^{(1)}$ and $v^{(2)}$ are both given by the (unique) solution v to the problem when the body rotates in an infinite homogeneous fluid, since $\partial v/\partial\theta = 0$ when $\theta = \pi/2$. Consequently the solution for the velocity fields in either of the fluids is the same as if the two fluids were homogeneous; the velocity fields are therefore independent of the viscosities and the planar interface between the fluids is stress-free.

The foregoing result is a particular case of a wider class of two-fluid flows. What is important is that there should be *geometrical* symmetry of the system about the planar interface as well as the axis of rotation. The fact that the body straddles the interface is not significant. Thus for a system of rigid bodies which have an axis of symmetry and also possess reflection symmetry about the planar interface between two immiscible fluids which is perpendicular to the axis of symmetry of the system, one can again say that if the system rotates with angular velocity Ω about the axis of symmetry, the Stokes flow generated in either of the fluids is independent of the viscosities and is just the solution appropriate to the rotation of the system of bodies with the same angular velocity in a homogeneous fluid.

A further generalization of the result can be made to an infinite system of rigid bodies and immiscible fluids which are stratified in layers perpendicular to the axis of symmetry of the system of bodies. Provided that the system of bodies possesses reflection symmetry about any of the planar interfaces, which are therefore equally spaced, then the solution of Stokes' equations for the multi-fluid flow will be the same as if the system of bodies were rotating in an infinite homogeneous fluid. The velocity field for the homogeneous fluid flow is a periodic function of z which is even about any one of the planes of the interfaces in the multi-fluid flow problem. Thus the stress continuity condition as well as the velocity continuity condition at any interface for the multi-fluid flow is automatically satisfied.

It is therefore evident that if in a Stokes flow of a homogeneous fluid, there is a subregion τ of the fluid which is entirely surrounded by the rest of the fluid and whose boundary is stress-free, then the viscosity of the fluid within τ can be varied arbitrarily without violating the continuity conditions on the velocity and stress across their boundary. It would appear that since in the foregoing examples the property of reflection symmetry of the system about an interface is crucial, a similar result does not occur in multi-fluid systems with non-planar interfaces. However non-planar interfaces can of course be at rest and in the next section we give a result pertaining to such multi-fluid flows.

5. MULTI-FLUID FLOWS WITH STATIC INTERFACES

We now suppose that viscous fluid is contained in the region between non-intersecting axially symmetric rigid surfaces S_1 and S_2 which rotate about their common axis of symmetry with constant angular velocities Ω_1 and $-\Omega_2$ respectively, with $\Omega_1, \Omega_2 > 0$. If the fluid motion is a Stokes flow, $\mathbf{v} = v(R, z)\hat{\boldsymbol{\phi}}$ where $v(R, z)$ is a solution of

$$L^2[v] \equiv \frac{\partial^2 v}{\partial R^2} + \frac{1}{R}\frac{\partial v}{\partial R} - \frac{v}{R^2} + \frac{\partial^2 v}{\partial z^2} = 0 \qquad [15]$$

satisfying the boundary conditions

$$v = \Omega_1 R \text{ on } c_1, \quad v = -\Omega_2 R \text{ on } c_2, \qquad [16]$$

where $c_i(i = 1, 2)$ is a meridian section of S_i. The solution of this boundary value problem can be formally expressed in terms of the Green's function as follows:

$$v(R, z) = \frac{1}{4\pi}\sum_{i=1}^{2}\int_{c_i} R'\Omega_i\left\{\int_0^{2\pi} e^{i(\phi'-\phi)}\frac{\partial G}{\partial n_i}\,\mathrm{d}\phi'\right\}\mathrm{d}l_i, \qquad [17]$$

where the Green's function is defined as the solution of

$$\nabla^2 G = -4\pi\delta(\mathbf{r}' - \mathbf{r})$$

subject to $G = 0$ on S_i for $i = 1, 2$.

If P_0 is any point on the axis of symmetry within S_1 and $\mathscr{L}$ is a straight line through P_0 which intersects S_1 and S_2 in P_1, P_2 respectively, it is clear that $v(R, z)$ is continuous along $\mathscr{L}$ and is positive at P_1 and negative at P_2 except when $\mathscr{L}$ coincides with the axis of symmetry. Hence there exists at least one point P of $\mathscr{L}$ at which $v(R, z) = 0$. As the orientation of $\mathscr{L}$ changes, the point P will trace out a surface S in the fluid with the property that the velocity vanishes over S.

Assuming that in general such a surface S exists, we now consider the two-fluid flow of immiscible fluids with viscosities $\mu_i(i = 1, 2)$, the fluid with viscosity μ_i occupying the region bounded by S_1 and S and the fluid with viscosity μ_2 occupying the region between S and S_2. We now suppose that S_1 and S_2 rotate about the axis of symmetry with angular velocities $(\mu_2/\mu_1)\Omega_1$ and $-\Omega_2$ respectively. The equations governing the motions of the fluids will be satisfied if the pressure $p = 0$ and the velocity $\mathbf{q}$ is given by

$$\mathbf{q} = (\mu_2/\mu_1)v(R, z)\hat{\boldsymbol{\phi}} \quad \text{between} \quad S_1 \text{ and } S,$$

and

$$\mathbf{q} = v(R, z)\hat{\boldsymbol{\phi}} \text{ between } S \text{ and } S_2,$$

where $v(R, z)$ is given by [17]. The velocity and stress continuity conditions are satisfied automatically at the static interface. Furthermore the torque on either S_1 or S_2 is the same as if the whole region between S_1 and S_2 is filled by a homogeneous fluid of viscosity μ_2 and S_1 rotates with angular velocity Ω_1.

A particular example would be if S_1 is the sphere $r = a$ and S_2 is the sphere $r = b(b > a)$. The solution of [16] reduces to

$$v = [-(\Omega_1 a^3 + \Omega_2 b^3)r/(b^3 - a^3) + a^3 b^3(\Omega_1 + \Omega_2)/r^2(b^3 - a^3)]\sin\theta \qquad [18]$$

with (r, θ, ϕ) denoting spherical polar coordinates. The surface S is given by

$$r^3 = a^3 b^3(\Omega_1 + \Omega_2)(\Omega_2 b^3 + \Omega_1 a^3)^{-1}. \qquad [19]$$

The torques acting on S_1 and S_2 are opposite in direction and of equal magnitude

$$8\pi\mu_2(\Omega_1 + \Omega_2)a^3b^3/(b^3 - a^3). \qquad [20]$$

6. TORQUE REDUCTION PRODUCED BY A VISCOSITY DISCONTINUITY

In this problem we consider an axially symmetric body which rotates about its axis of symmetry with small angular velocity in a system of two immiscible fluids, one of which extends to infinity. The detailed calculations we shall give are for the case when the body is a sphere, but the analysis can be applied to the class of axially symmetric bodies considered by Jeffrey (1915) or, we may conjecture, to any axially symmetric body which, when rotating in an infinite homogeneous fluid, has stream surfaces which are similar to the body shape.

A sphere of radius a rotates with angular velocity Ω in immiscible fluids of viscosities μ_1 and μ_2. The fluid of viscosity μ_1 is confined to the region $a \le r \le c$ and the rest of space is filled by the fluid of viscosity μ_2. With the notation that the suffix i applies to the region occupied by the fluid with viscosity μ_i, the flows in the two fluids will be given by

$$\mathbf{q}_i = v_i(R, z)\hat{\boldsymbol{\phi}}, p_i = 0, \qquad [21]$$

where $v_i(R, z)$ satisfies [16], the boundary conditions

$$v_1 = v_2, \mu_1 \frac{\partial}{\partial R}\left(\frac{v_1}{R}\right) = \mu_2 \frac{\partial}{\partial R}\left(\frac{v_2}{R}\right) \qquad [22]$$

on $r = c$ together with the conditions

$$\left.\begin{aligned} v_1 &= \Omega R, \quad (r = a) \\ v_2 &= 0, \quad (r = \infty). \end{aligned}\right\} \qquad [23]$$

The solutions of [16] satisfying [22] and [23] are

$$v_1 = \Omega r\left\{1 - \frac{1}{a^3}\left[\frac{\mu_1}{\mu_2 c^3} + \frac{1}{a^3} - \frac{1}{c^3}\right]^{-1} + \frac{1}{r^3}\left[\frac{\mu_1}{\mu_2 c^3} + \frac{1}{a^3} - \frac{1}{c^3}\right]^{-1}\right\}\sin\theta \quad (a \le r \le c) \qquad [24]$$

and

$$v_2 = \frac{\mu_1 \Omega \sin\theta}{\mu_2 r^2}\left[\frac{\mu_1}{\mu_2 c^3} + \frac{1}{a^3} - \frac{1}{c^3}\right]^{-1} \quad (r \ge c). \qquad [25]$$

The torque resisting the rotation of the sphere is

$$T_1 = 8\pi\mu_1\Omega\lambda\left[\frac{1}{c^3} + \lambda\left(\frac{1}{a^3} - \frac{1}{c^3}\right)\right]^{-1},$$

where $\lambda = \mu_2/\mu_1$. Now if the entire region $r > a$ were occupied by homogeneous fluid with viscosity μ_2, the torque acting on the sphere would be

$$T_2 = 8\pi\mu_2\Omega a^3.$$

Thus

$$T_1 = \gamma T_2$$

where

$$\gamma = [s^3 + \lambda(1 - s^3)]^{-1}, \qquad [26]$$

with $s = a/c < 1$. Clearly $\gamma < 1$ when $\lambda > 1$. Thus the torque acting on the sphere is always reduced when there is a layer of fluid adjacent to the sphere with a viscosity lower than that of the bulk of the fluid. The reduction of the torque can be substantial. For instance if $\lambda = 10$, then $\gamma \approx 0.27$ when $s = 0.9$ and $\gamma \approx 0.11$ when $s = 0.5$. A two-dimensional analogue of this flow exists when the body is a circular cylinder and in this case the Navier–Stokes equations can be solved exactly.

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Résumé On montre que la solution des équations de Stokes pour ùn corps axi-symétrique en rotation, symétrique de plus par rapport au plan de l'interface entre deux fluides visqueux, au repos, non miscibles et s'étendant à l'infini, est indépendante des viscosités des fluides, et identique à la solution correspondant à des fluides de même viscosité. On généralise ce résultat au cas d'un système axi-symétrique en rotation, de corps possédant chacun comme plan de symétrie une interface plane dans un ensemble de fluides stratifiés. On obtient également un résultat analogue pour des systèmes à deux fluides à interface stationnaire non plane.

On étudie l'effet de réduction de couple produit par la présence d'une couche d'un second fluide, adjacente à un corps axi-symétrique en rotation, et on donne les calculs explicites dans le cas d'une sphère. On donne les preuves d'unicité pour les écoulements de Stokes multifluides généraux, dans des systèmes finis ou infinis et on détermine la structure asymptotique loin du corps du champ de vitesse, pour un écoulement axi-symétrique provoqué par la rotation ou la translation d'un corps axi-symétrique dans un fluide homogène.

Auszug—Die Loesung der Stokesschen Gleichungen fuer einen sich drehenden Rotationskoerper mit Reflektionssymmetrie in Bezug auf eine ebene Grenzflaeche zwischen zwei unendlichen nicht mischbaren ruhenden zaehen Fluessigkeiten wird betrachtet. Sie ist von den Viskositaeten der Fluessigkeiten unabhaengig, und mit der Loesung fuer Fluessigkeiten gleicher Viskositaet identisch. Das Ergebnis wird auf ein rotierendes axisymmetrisches System von Koerpern verallgemeinert, das Reflektionssymmetrie in Bezug auf jede Grenzflaeche eines ebenen Systems geschichteter Fluessigkeiten besitzt. Ebenso wird ein analoges Ergebnis fuer Zweifluessigkeitssysteme mit nicht ebener statischer Grenzflaeche entwickelt. Die Verringerung des Drehmoments durch die Anwesenheit einer zweiten, den sich drehenden Rotationskoerper umschliessenden, Fluessigkeitsschicht wird betrachtet und fuer den Fall einer Kugel explizit berechnet. Die Eindeutigkeit der Loesung fuer allgemeine Stokessche Stroemungen mehrerer Fluessigkeiten in endlichen und unendlichen Systemen wird bewiesen, und es wird die asymptotische Fernfeldstruktur des Geschwindigkeitsfelds einer axisymmetrichen Stroemung bestimmt, die durch Drehung oder Verschiebung eines Rotationskoerpers in einer homogenen Fluessigkeit hervorgerufen wird.

Резюме—Вданной работе приводится решение стоксовых уравнений для вращающегося осесимметричного тела, обладающего взаимной симметрией относительно плоской поверхности раздела между двумя бесконечно—простирающимися несмешивающимися покоящимися вязкими жидкостями: это решение не зависит от вязкости жидкостей и тождественно решению для случая, когда жидкости имеют одинаковую вязкость. Результат обобщен для вращающейся осесимметричной системы тел, обладающих взаимной симметрией относительно каждой поверхности раздела плоско—расслоенной совокупности жидкостей. Аналогичный результат устано влен также для системы двух жидкостей с неплоской поверхностью раздела. Учтено понижение момента вращения, вызываемое присутствием второго слоя жидкости, примыкающего к вращающемуся осесимметричному телу, и приведены подробные расуеты для случая шара. Проверена единственность решения для обобщенных многофазных стоксовых течений в конечных и бесконечных системах и определена асимптотически уделенная структура поля скоростей для осесимметричного течения, вызываемого вращением или перемещением осесимметричного тела в однородной жидкости.